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Classification of G-invariant configurations of Einstein-Cartan theory on a multidimensional universe

GERD RUDOLPH

Sektion Physik und Naturwissenschaftlich-Theoretisches Zentrum
der Karl-Marx-Universität Leipzig,
7010 Leipzig, Karl-Marx-Platz 10/11, DDR

Abstract. *G-invariant configurations (τ, γ) of Einstein-Cartan theory on a multidimensional universe M are shown to be in 1 - 1-correspondence with a quintuplet $(\tilde{\tau}, \xi, \hat{\alpha}, \phi, \Psi)$ of geometrical objects living on bundles over physical spacetime $\tilde{M} = M/G$. Moreover, explicit formulae for τ are given in terms of classifying objects.*

1. INTRODUCTION

During the last years a lot of work on dimensional reduction of field theories has been done. At one hand dimensional reduction of pure gauge theories has been investigated see, [1], [2], [3], [4] and references therein. On the other hand Kaluza-Klein theories (dimensional reduction of gravity) have been extensively studied, both from a more intuitive and from a mathematical (geometrical) point of view, for the latter see [5], [6] and references therein. In both cases the starting point is the same. One has a multidimensional universe M with a symmetry group G acting on M in a sufficiently regular way and considers a field theory on M, whose configurations are supposed to be G-invariant. Then one has to solve two problems:

1. Classification of G-invariant configurations,
2. Reduction of the action – due to G-invariance – to an action on physical

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space time $\widetilde{M} = M/G$.

Doing this one obtains interesting unification schemes in both above mentioned cases. Obviously, one can also combine the two cases and consider Einstein-Yang-Mills systems [7].

In this paper we start to investigate dimensional reduction of a generalized gravitational theory, namely Einstein-Cartan theory. For an excellent analysis of this theory see [8]. As far as we know, up to now only special cases of theories of this type have been studied within the dimensional reduction scheme, see for example [9]. Thus, the main point of this paper is to give a full classification of G-invariant configurations of Einstein-Cartan theory on a multidimensional universe.

A second point is the following: In the geometrical approach to Kaluza-Klein theories as developed in [5] one has a clear understanding of the structures one deals with, but, finally, all calculations are done using local n-bein-techniques. In this paper we show that similar techniques as developed in [3] for gauge theories can be used here, and that all local considerations can be avoided. For that purpose we use the gauge theoretical formulation of gravity in the spirit of [10].

2. EINSTEIN-CARTAN THEORY ON A MULTIDIMENSIONAL UNIVERSE

Let M be a n-dimensional manifold and G a compact, connected Lie group acting differentiably to the left on M:

$$M \times G \ni (x, g) \longrightarrow \delta(x, g) \in M. \tag{1.1}$$

For clearness of presentation we assume G to act without fix point, thus $M(\widetilde{M}, G)$ is a principal bundle over the orbit space $\widetilde{M} = M/G$ with typical fibre G, (the right action of G on M is defined by $\widetilde{\delta}_g := \delta_{g^{-1}}$).

The generalization to the case when the typical fibre is a homogeneous space is straightforward.

A configuration of Einstein-Cartan theory on M is a pair (τ, γ), where γ is a (pseudo)-Riemannian metric on M and τ is a linear connection on M compatible with γ:

$$D\gamma = 0. \tag{1.2}$$

We will treat τ as a connection form in the reper bundle LM and γ as a $GL(n, R)$-equivariant mapping

$$\gamma : LM \longrightarrow (R^n)^* \overset{s}{\otimes} (R^n)^*. \tag{1.3}$$

It is well known that the action of G lifts naturally to the bundle space LM. We denote this lift by

(1.4) $$LM \times G \ni (e, g) \longrightarrow \sigma(e, g) \in LM,$$

with $\sigma_g \in \mathrm{Aut}\,(LM)$, $\sigma_g(e) \equiv \sigma(e, g)$.

Now we can define what we mean by a G-invariant configuration:

(1.5a) $$\sigma_g^* \tau = \tau,$$

(1.5b) $$\sigma_g^* \gamma = \gamma.$$

Our aim is to classify pairs (τ, γ) satisfying (1.2) and (1.5).

3. BUNDLE REDUCTIONS AND SPLITTINGS

Suppose we are given a configuration (τ, γ) satisfying (1.2) and (1.5).

PROPOSITION 3.1. *There is a sequence of bundle reductions defined by γ and the action of G:*

(2.1) $$LM \longrightarrow OM \longrightarrow \widetilde{OM} \longrightarrow \widehat{OM},$$

where OM is the bundle of orthonormal frames over M, $\widetilde{OM}$ a principal bundle with typical fibre $O(m) \times O(n-m)$ over M and $\widehat{OM}$ a principal bundle with typical fibre $O(m)$ over M.

Proof. The first reduction is standard [11], [12]:

(2.2) $$OM := \{e \in LM : \gamma(e) = \eta\},$$

where $\eta \in (R^n)^* \overset{s}{\otimes} (R^n)^*$ is in the standard basis of R^n given by $\eta = \mathrm{diag}\,(-1, +1, \ldots, +1)$. The second reduction is defined by a splitting of the tangent bundle:

(2.3) $$TM = V \oplus H,$$

where V is the canonical vertical distribution defined by the right group action,

(2.4) $$V_x := \tilde{\delta}'_x(\mathfrak{G}), \qquad x \in M,$$

$\mathfrak{G}$-Lie algebra of G, and H is the to V with respect to γ orthogonal complement. Splitting (2.3) is a section of the associated bundle $E = OM \times_{O(n)} G_{m,n}$, with $G_{m,n} = O(n)/(O(m) \times O(n-m))$ being the space of orthogonal with respect to η decompositions of R^n. Treating this section as an equivariant mapping $\zeta : OM \longrightarrow G_{m,n}$ and fixing one decomposition ζ_0, given by

(2.5a) $$R^n = R^m \oplus R^{n-m},$$

we put

$$\widetilde{OM} := \{e \in OM : \zeta(e) = \zeta_0\}. \tag{2.5b}$$

(For simplicity of notation we write $O(n)$ and $O(m)$ instead of $O(n-1, 1)$ and $O(m-1, 1)$!).

The last reduction is not essential, but convenient. First notice that the canonical projection $O(m) \times O(n-m) \longrightarrow O(n-m)$ induces a surjective bundle homomorphism

$$f : \widetilde{OM}(O(m) \times O(n-m), M) \longrightarrow \widetilde{\widetilde{OM}}(O(n-m), M). \tag{2.6}$$

Now fix a basis $(\hat{\epsilon}_a)$ in $\mathfrak{G}$, take the corresponding fundamental vector fields $(\epsilon_a)_x := \tilde{\delta}'_x(\hat{\epsilon}_a)$ and perform a standard orthonormalization procedure. The result is a section s in $\widetilde{\widetilde{OM}}$. We put

$$\widehat{OM} := \{e \in \widetilde{OM} : f(e) = s(\rho(e))\}, \tag{2.7}$$

where ρ is the canonical projection in $\widetilde{OM}$. ∎

PROPOSITION 3.2. *There is a natural splitting*

$$T\widehat{OM} = \hat{V} \oplus \hat{H}, \tag{2.8}$$

induced by γ and by the action of G on $\widehat{OM}$.

Proof. First notice that $\widehat{OM}$ is G-invariant, and, therefore, it is a principal G--bundle over $O\widetilde{M}$, the bundle of orthonormal (with respect to $\tilde{\gamma}$ – the metric induced by the G-invariant metric γ on $\widetilde{M}$) frames on $\widetilde{M}$. The free, right action of G on $\widehat{OM}$ is defined by $\tilde{\sigma}_g := \sigma_{g^{-1}}$. (We denote the restriction of group actions to subbundles by the same letter). Since σ is the lift of δ we have the following commutative diagram:

$$\begin{array}{ccc} \widehat{OM} & \xrightarrow{\chi_1} & O\widetilde{M} \\ \pi_1 \downarrow & & \downarrow \chi_2 \\ M & \xrightarrow{\pi_2} & \widetilde{M} \end{array} \tag{2.9}$$

Now we put

$$\hat{V}_e := \tilde{\sigma}'_e(\mathfrak{G}) \tag{2.10}$$

and

$$\hat{H}_e := (\pi'_1)^{-1}(H_{\pi_1(e)}). \tag{2.11}$$

Obviously, $\hat{V}$ and $\hat{H}$ are differentiable distributions spanning $T\widehat{OM}$. ∎

COROLLARY 3.1. *The splitting (2.8) defines a G-connection in the bundle* $\widehat{OM} \longrightarrow \widetilde{OM}$.

Proof. Obvious. ∎

We denote the corresponding G-connection form on $\widehat{OM}$ by ξ and observe that ξ is the pull-back under π_1 of the G-connection form defined by the splitting (2.3).

4. CLASSIFICATION OF G-INVARIANT CONFIGURATIONS

For a given pair (τ, γ) satisfying (1.2) and (1.5) we have the sequence of reductions (2.1) and the G-connection (2.8). It is known that a linear connection on LM is reducible to OM if and only if (1.2) is fulfilled [12].

Restricted to OM (1.2) just means that τ is $o(n)$-valued. Thus, τ fulfilling (1.2) is completely given by its values on OM, and – obviously – also by its values on the subbundle $\widehat{OM}$.

On the other hand γ is constant on OM, so that the classification problem is now reduced to characterizing the restriction of τ to $\widehat{OM}$. For that purpose we use the decomposition [11]

$$\tau = \omega + \alpha, \tag{3.1}$$

where ω is the Levi-Civita connection form corresponding to γ and α is a tensorial 1-form of type $\mathrm{Ad}\,(Gl(n, R))$ with values in $gl(n, R)$. We shall denote the restrictions of ω and α to $\widehat{OM}$ by the same letters. Notice that (1.2) and (1.5) imply

$$\sigma_g^*\omega = \omega, \tag{3.2a}$$

$$\sigma_g^*\alpha = \alpha. \tag{3.2b}$$

PROPOSITION 4.1. *The 1-form α is completely characterized by*

1. *A Mapping*

$$\widehat{OM} \ni e \longrightarrow \Psi(e) \in \mathfrak{G}^* \otimes o(n), \tag{3.3a}$$

satisfying

$$\tilde{\sigma}_g^*\Psi = \mathrm{Ad}'g^{-1}(\Psi), \tag{3.3b}$$

where Ad′ *denotes the coadjoint representation of G in* $\mathfrak{G}^*$.

2. *A $o(n)$-valued 1-form on* $\widetilde{OM}$

$$T_{\tilde{e}}\widetilde{OM} \ni X \longrightarrow \tilde{\alpha}_{\tilde{e}}(X) \in o(n). \tag{3.4}$$

Proof. α is completely characterized by its values on $\widehat{OM}$. We decompose α on $\widehat{OM}$ due to (2.8) and put:

$$\Psi(e)(A) := (\alpha_{\hat{V}})_e(\tilde{\sigma}'_e(A)), \qquad A \in \mathfrak{G}, \tag{3.5}$$

$$\tilde{\alpha}_{x_1(e)}(X) := (\alpha_{\hat{H}})_e(X^{\hat{H}}), \tag{3.6}$$

where $X^{\hat{H}}$ is the horizontal with respect to ξ lift of $X \in T_{\tilde{e}}\tilde{OM}$. To prove (3.3b) we use the G-invariance (3.2b) of α and the fact that

$$\tilde{\sigma}'_{\tilde{\sigma}_g(e)} = \tilde{\sigma}'_g \circ \tilde{\sigma}'_e \circ \operatorname{Ad} g, \tag{3.7}$$

which can be easily verified. Then we have

$$\begin{aligned} \Psi_{\tilde{\sigma}_g(e)}(A) &= (\alpha_{\hat{V}})_{\tilde{\sigma}_g(e)}(\tilde{\sigma}'_{\tilde{\sigma}_g(e)}(A)) = \\ &= (\alpha_{\hat{V}})_{\tilde{\sigma}_g(e)}(\tilde{\sigma}'_g \circ \sigma'_e \circ \operatorname{Ad} g(A)) = \\ &= (\tilde{\sigma}_g^* \alpha_{\hat{V}})_e(\tilde{\sigma}'_e \circ \operatorname{Ad} g(A)) = \\ &= \Psi_e(\operatorname{Ad} g(A)). \end{aligned}$$

We are left with proving that definition (3.6) is correct. But this is a simple consequence of (3.2b). ∎

It remains to analyze ω.

PROPOSITION 4.2. *On OM we have*

$$\begin{aligned} &\langle \vartheta(Z), \omega(Y)\, \vartheta(X)\rangle = \\ &= -\langle \vartheta(Y), \mathrm{d}\vartheta(Z, X)\rangle + \langle \vartheta(Z), \mathrm{d}\vartheta(X, Y)\rangle + \langle \vartheta(X), \mathrm{d}\vartheta(Y, Z)\rangle, \end{aligned} \tag{3.8}$$

where X, Y, $Z \in TOM$, ϑ is the canonical 1-form [11] *on OM and* $\langle\ ,\ \rangle$ *is the scalar product on R^n given by η.*

Proof. On OM ω is completely determined by the equation $\mathrm{d}\vartheta + \omega \wedge \vartheta = 0$, (vanishing of torsion).

(3.8) is obtained by solving this equation with respect to ω – a standard procedure which we omit here. ∎

Formula (3.8) says that ω is completely determined by ϑ and $\mathrm{d}\vartheta$. For ϑ we have:

PROPOSITION 4.3. *The soldering form ϑ is on $\widehat{OM}$ completely characterized by:*

1. *A mapping*

(3.9a) $$\widehat{OM} \ni e \longrightarrow \phi(e) \in \mathfrak{G}^* \otimes R^{n-m},$$

satisfying

(3.9b) $$\tilde{\sigma}_g^* \phi = \mathrm{Ad}' g^{-1}(\phi).$$

2. *The soldering form $\tilde{\vartheta}$ on $O\widetilde{M}$, with*

(3.9c) $$\vartheta_{\hat{H}} = x_1^* \tilde{\vartheta}.$$

Proof. ϑ is completely characterized by its values on $\widehat{OM}$. Due to (2.8) we put:

(3.10) $$\phi(e)(A) := (\vartheta_{\hat{V}})_e(\tilde{\sigma}_e'(A)), \qquad A \in \mathfrak{G},$$

(3.11) $$\tilde{\vartheta}_{\tilde{e}}(X) := (\vartheta_{\hat{H}})_e(X^{\hat{H}}),$$

where again $X^{\hat{H}}$ is the horizontal with respect to ξ lift of X.

Using the definition of ϑ and the fact that e is an adapted frame we have:

$$\phi(e)(A) = e^{-1} \circ \pi_1' \circ \tilde{\sigma}_e'(A) = e^{-1} \circ \tilde{\delta}'_{\pi_1(e)}(A) \in R^{n-m}.$$

Formula (3.9b) is shown in the same way as (3.3b). Definition (3.11) is correct because ϑ is G-invariant. (ϑ is invariant under natural lifts of arbitrary diffeomorphisms of M!).

It remains to show that $\tilde{\vartheta}$ coincides with the soldering form on $O\widetilde{M}$. Treating e and $\tilde{e}$ as mappings [11], $e : R^n \longrightarrow T_{\pi_1(e)}M$, $\tilde{e} : R^m \longrightarrow T_{x_2(e)}\widetilde{M}$, we have $e^{-1}\restriction_H = \tilde{e}^{-1} \circ \pi_2'$.

Using this and $x_2' \circ x_1' = \pi_2' \circ \pi_1'$, see (2.9), we have $\tilde{\vartheta}_{\tilde{e}}(x_1'(X)) = \tilde{e}^{-1} \circ x_2' \circ x_1'(X) = e^{-1} \circ \pi_1'(X) = (\vartheta_{\hat{H}})_e(X)$, for $X \in \hat{H}$. ∎

Now we want to calculate ω on $\widehat{OM}$. For that purpose we use (2.8) and the following natural – with respect to (2.5a) – decomposition of the Lie algebra $o(n)$:

(3.12a) $$o(n) = o(m) \oplus o(n-m) \oplus \mathfrak{M},$$

where

(3.12b) $$\mathfrak{M} := \left\{ \left[\begin{array}{c|c} O & A \\ \hline -A^T & O \end{array} \right] ; A \in L(R^m, R^{n-m}) \right\}.$$

Being $o(n)$-valued ω has three components, $\omega^1 \equiv \omega^{o(m)}$, $\omega^2 \equiv \omega^{o(n-m)}$ and $\omega^3 \equiv \omega^{\mathfrak{M}}$. Moreover, we will use the following:

LEMMA 4.1. *There exist canonical isomorphisms*

(3.13a) $i_e : (\hat{H}_e')^* \longrightarrow (R^m)^*,$

(3.13b) $j_e : \overset{2}{\wedge}(\hat{H}_e')^* \longrightarrow o(m),$

where $(\hat{H}_e')^*$ *is the vector space dual to* $\hat{H}_e' = \hat{H}_e/\ker \pi_1'$.

Proof. There exists a canonical (horizontal with respect to the Levi-Civita connection $\tilde{\omega}$ on $O\tilde{M}$) vector field $\tilde{\mathfrak{z}}$ (with values in $(R^m)^*$) on $O\tilde{M}$, such that $\tilde{\vartheta}(\tilde{\mathfrak{z}}) = \mathrm{id}_{L(R^m, R^m)}$.

This field defines an isomorphism of vector space:

(3.14) $\tilde{\mathfrak{z}}: (\tilde{H}_{\tilde{e}})^* \longrightarrow (R^m)^*,$

where $\tilde{H}_{\tilde{e}}$ is the horizontal with respect to $\tilde{\omega}$ subspace of $T_{\tilde{e}}O\tilde{M}$. Taking the horizontal (with respecto to ξ) lift of $\tilde{H}_{\tilde{e}}$ to $\hat{H}_e$, we get an isomorphism $\tilde{H}_{\tilde{e}} = \hat{H}_e'$ which implies $(\tilde{H}_{\tilde{e}})^* \cong (\hat{H}_e')^*$. Combining this with (3.14), we get the isomorphism i. (3.13b) is obtained by taking $j := \tilde{\eta}^{-1} \circ \overset{2}{\wedge} i$, $\tilde{\eta} = \mathrm{diag}\,(-1, 1, \ldots, 1)$ being the Minkowski metric on R^m. ■

PROPOSITION 4.4. *The decomposition due to (2.8) of the* $o(m)$*-component* ω^1 *yields:*

(3.15) 1. $\omega^1_{\hat{H}} = x_1^* \tilde{\omega} \in \hat{H}^* \otimes o(m),$

with $\tilde{\omega}$ *being the Levi-Civita connection on* $O\tilde{M}$.

(3.16) 2. $\tilde{\sigma}^* \omega^1_{\hat{V}} = -(\phi^*\eta) \circ j(\Xi) \in \overset{0}{\wedge}\, \widehat{O\tilde{M}} \otimes o(m) \otimes \mathfrak{G}^*,$

where $\phi^*\eta : \mathfrak{G} \longrightarrow \mathfrak{G}^*$ *is the by* ϕ *induced scalar product on* $\mathfrak{G}$ *and* Ξ *is the curvature form of* ξ.

Proof. The first point is obvious because of (3.9c). Using the known fact [11] that

(3.17) $\mathrm{ver}\,[Z, X]_e = -2\,\tilde{\sigma}_e'(\Xi(Z, X)),$

where $Z, X \in \hat{H}$ and ver $(\cdot)$ means the vertical component with respect to ξ, formula (3.8) gives immediately:

(3.18) $\langle \vartheta_{\hat{H}}(Z), \omega^1_{\hat{V}}(Y)\, \vartheta_{\hat{H}}(X)\rangle = -\langle \vartheta_{\hat{V}}(Y), \phi(\Xi(Z, X))\rangle,\ Y \in \hat{V}.$

Inserting

$$\Xi(Z, X) = \Xi(i^* \circ \vartheta_{\hat{H}}(Z), i^* \circ \vartheta_{\hat{H}}(X)) = \langle \vartheta_{\hat{H}}(Z), j(\Xi)\, \vartheta_{\hat{H}}(X)\rangle$$

into (3.17) gives (3.16). ■

PROPOSITION 4.5. *The decomposition of* ω^2 *yields:*

$$(3.19)\qquad 1.\qquad \omega^2_{\hat{H}} = -1/2\,(D\phi\circ\phi^{-1} - (D\phi\circ\phi^{-1})^T) \in \hat{H}^*\otimes o(n-m),$$

where $D\phi = \mathrm{d}\phi + \mathrm{ad}'(\xi)\phi$, *with* ad′ *being the coadjoint representation of* $\mathfrak{G}$ *in* $\mathfrak{G}^*$.

$$(3.20)\qquad 2.\qquad \phi^*\eta\circ(\phi^{-1}\circ\tilde{\sigma}^*\omega^2_{\hat{V}}\circ\phi) = -1/2\{\phi^*\eta\circ\mathrm{ad} - (\phi^*\eta\circ\mathrm{ad})^{T_{13}} + (\phi^*\eta\circ\mathrm{ad})^{T_{12}}\} \in \overset{0}{\wedge}\,\widehat{OM}\otimes\mathfrak{G}^*\otimes\mathfrak{G}^*\otimes\mathfrak{G}^*,$$

where for $A, B, C \in \mathfrak{G}$, $\phi^*\eta\circ\mathrm{ad}\,(A,B,C) = \langle\phi(A), \phi([B,C])\rangle$ *and* $(\phi^*\eta\circ\phi^{-1}\circ\tilde{\sigma}^*\omega^2_{\hat{V}}\circ\phi)(A,B,C) = \langle\phi(A), \tilde{\sigma}^*\omega^2_{\hat{V}}(B)\phi(C)\rangle$, T_{ij} *are transpositions in the tensor product* $\mathfrak{G}^*\otimes\mathfrak{G}^*\otimes\mathfrak{G}^*$.

Proof. 1. Taking $Y \in \hat{H}$ and $X, Z \in \hat{V}$, generated by $A, B \in \mathfrak{G}$, we get from (3.8):

$$(3.21)\qquad \langle\phi(B), \omega^2_{\hat{H}}(Y)\phi(A)\rangle = -1/2\langle\phi(B), Y(\phi(A))\rangle + 1/2\langle\phi(A), Y(\phi(B))\rangle$$

But $Y(\phi(A)) = Y(\phi)(A) = D\phi(Y)(A)$, by the definition of the covariant derivative [11]. Inserting this into (3.21) gives (3.19).

2. For $X, Z, Y \in \hat{V}$, generated by $A, B, C \in \mathfrak{G}$, we have from (3.8)

$$(3.22)\qquad \langle\phi(B), \tilde{\sigma}^*\omega^2_{\hat{V}}(C)\phi(A)\rangle = -1/2\,\{\langle\phi(B), Y(\phi(A))\rangle - \langle\phi(A), Y(\phi(B))\rangle - \langle\phi(C), X(\phi(B))\rangle\}.$$

But

$$Y(\phi(A))_e = \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\,\phi(\tilde{\sigma}_{\exp tC}(e))(A) = \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\,\phi_e(\mathrm{Ad}\,(\exp tC)(A)) = \mathrm{ad}'C(\phi)(A).$$

Inserting this into (3.22) gives (3.20). ■

Remark 4.1. The right-hand-side of (3.20) is the Levi-Civita connection of the metric on G induced by the scalar product $\phi^*\eta$ on $\mathfrak{G}$.

Of course, in the general case those connections will be different for each orbit of G on M.

PROPOSITION 4.6. *The decomposition of* ω^3 *yields:*

(3.23) 1. $\omega^3_{\hat{H}} = -\phi \circ (i \otimes \mathrm{id}) \circ \Xi \in \hat{H}^* \otimes (R^m)^* \otimes R^{n-m}$,

(3.24) 2. $\tilde{\sigma}^* \omega^3_{\hat{V}} = 1/2 \{ i \circ D\phi \circ \phi^{-1} + (i \circ D\phi \circ \phi^{-1})^T \} \circ$
$\circ \phi \in \overset{0}{\Lambda} \widehat{OM} \otimes (R^m)^* \otimes R^m$.

(In fact we have written down only the $L(R^m, R^{n-m})$-component of ω^3, see (3.12b)).

Proof. 1. Taking $X, Y \in \hat{H}$ and $Z \in \hat{V}$, generated by $A \in \mathfrak{G}$, we get from (3.8)

(3.25) $$\langle \phi(A), \omega^3_{\hat{H}}(Y)\, \vartheta_{\hat{H}}(X) \rangle = -1/2 \langle \phi(A), \vartheta_{\hat{V}}([X, Y]) \rangle.$$

But $\vartheta_{\hat{V}}([X, Y]) = \phi((i \otimes \mathrm{id}) \circ \Xi(\vartheta_{\hat{H}}(X), Y))$.

2. Taking $X \in \hat{H}$, Y, $Z \in \hat{V}$, generated by B, $A \in \mathfrak{G}$, we have from (3.8):

$$\langle \phi(A), \tilde{\sigma}^* \omega^3_{\hat{V}}(B)\, \vartheta_H(X) \rangle = 1/2 \{ \langle \phi(A), D\phi(X)(B) \rangle + \langle \phi(B), D\phi(X)(A) \rangle \}.$$

To obtain from this (3.24) we use

$$D\phi(X)(A) = D\phi(i^* \vartheta_{\hat{H}}(X))(A) = (i \circ D\phi)(A)\, \vartheta_H(X). \quad ■$$

Propositions 4.4, 4.5 and 4.6 give explicit formulae for the Levi-Civita-connection-part of τ, which will be necessary for investigating field dynamics. Moreover, summarizing Cor. 3.1, Propositions 4.1, 4.3, 4.4, 4.5 and 4.6 we obtain the

CLASSIFICATION THEOREM. *A G-invariant Einstein-Cartan configuration (τ, γ) is in 1-1-correspondence with a quintuplet of geometrical objects $(\tilde{\tau}, \xi, \hat{\alpha}, \phi, \Psi)$, where $\tilde{\tau}$ is the induced Einstein-Cartan configuration on $O\tilde{M}$, ξ a G-principal connection in $\widehat{OM} \longrightarrow O\tilde{M}$, $\hat{\alpha} \in \overset{\circ}{\Lambda} O\tilde{M} \otimes (o(n-m) \oplus \mathfrak{M})$ and ϕ and Ψ are vector-space valued, G-equivariant functions on $\widehat{OM}$ defined by* (3.10) *and* (3.5).

Proof. The only point which remained to show is how $\tilde{\tau}$ is obtained. We decompose $\tilde{\alpha}$ (see (3.4)) due to (3.12a) and put: $\tilde{\tau} = \tilde{\omega} + \tilde{\alpha}^{o(m)}$. The remaining two components of $\tilde{\alpha}$ give $\hat{\alpha}$. ■

Remark 4.2. A priori classifying configurations of two different G-invariant configurations are living on different reduced bundles, because every γ gives an individual $\widehat{OM}$. However, we may distinquish one, say (τ_0, γ_0), and – by a vertical automorphism β of LM – relate every configuration (τ, γ) to γ_0 by taking the gauge-equivalent configuration (τ', γ_0), with $\tau' = \beta^* \tau$.

In that way all G-invariant configurations will be classified in terms of objects living on the same reduced bundle $\widehat{OM}$ (resp. $O\widetilde{M}$) defined by γ_0.

In a next paper we shall analyze torsion and curvature of G-invariant Einstein-Cartan configurations and discuss dynamical aspects.

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